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ON HOMOGENEOUS STRONGLY α-n-IRREDUCIBLE IDEALS OF COMMUTATIVE GRADED RINGS

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Abstract. Let M be a commutative additive monoid with identity element denoted by 0 and R be a commutative graded ring, which is graded by M. Let I be a homogeneous ideal of R, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function, where $\mathcal{L}(R)$ is the set of all homogeneous ideals of R. In this paper, we generalize the notion of strongly α-n-irreducible ideals to the context of graded rings, that is, if whenever $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$ for homogeneous ideals $I_1, \ldots, I_{n+1}$ of R, there are n of the I_i's whose intersection is in I. We study the transfer of this new concept in the idealization of graded modules and the amalgamation of graded rings along a homogeneous ideal.

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1. Introduction

In this paper, all rings under consideration are assumed to be commutative with nonzero identity and all modules are assumed to be nonzero unital. R will always represent such a ring, and E will represent such an R-module. Also, M will represent a commutative additive monoid with identity element denoted by 0. By a graded ring R, we mean a ring graded by M, i.e., a direct sum of subgroups R_α of R such that $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$ for every α, $\beta \in G$. A nonzero element $x \in R$ is called homogeneous if it belongs to one of the R_α; homogeneous of degree α if $x \in R_\alpha$. The set $h(R) = \bigcup_{\alpha \in G} R_\alpha$ is the set of homogeneous elements of R. An ideal I of R is said to be a homogeneous ideal if the homogeneous components of every element of I belong to I, equivalently, if I is generated by homogeneous elements. In the literature, homogeneous ideals and graded ideals are used interchangeably, but homogeneous ideals seem to be preferred, so we will use that term in this paper. Let I be a homogeneous ideal. Then I is called a homogeneous prime ideal if whenever $xy \in I$ for some $x, y \in h(R)$, we have $x \in I$ or $y \in I$. Note that if M is a torsionless monoid (that is, if M is cancellative and the group G of differences of M is a torsion-free abelian group, i.e.,

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$G = \{a - b \mid a, b \in M\}$), then I is homogeneous prime if and only if I is a prime ideal. Recently, there have been various generalizations of homogeneous prime ideals in several papers. Among the many recent generalizations of the notion of homogeneous prime ideals in the literature, we find the following, first defined by M. Refai and K. F. Al Zoubi [18, Definition 2.13]. A proper homogeneous ideal I of a commutative graded ring R is said to be homogeneous irreducible if $I = J \cap K$ for some homogeneous ideals I and K of R implies that either $I = J$ or $I = K$. A proper ideal I of R is said to be strongly homogeneous irreducible if for any homogeneous ideals J, K of R, $J \cap K \subseteq I$ implies that $J \subseteq I$ or $K \subseteq I$. According to [12, Definition 2.1], a proper homogeneous ideal I of R is called a homogeneous n-absorbing ideal if whenever $x_1 \cdots x_{n+1}$ for $x_1, \ldots, x_{n+1} \in h(R)$, then there are n of the x_i's whose product is in I. Thus a homogeneous 1-absorbing ideal is just a homogeneous prime ideal.

In [20], Zeidi defined an ideal I to be a strongly n-irreducible ideal of R if whenever $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \subseteq I$ for any $I_1, I_2, \ldots, I_{n+1}$ ideals of R, there are n of the I_i's whose intersection is contained in I. In [5], the authors generalized this concept to the context of graded rings by extending several results proved in [20]. Let n be a positive integer. According to [5], a proper homogeneous ideal I of a graded ring R is called strongly homogeneous n-irreducible if for any homogeneous ideals $I_1, I_2, \ldots, I_{n+1}$ of R, $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \subseteq I$ implies that there are n of the I_i's whose intersection is in I. Obviously, every strongly homogeneous irreducible ideal is a strongly homogeneous n-irreducible ideal.

The set of ideals of R will be denoted by $\mathcal{L}(R)$. By a proper ideal I of R we mean an ideal $I \in \mathcal{L}(R)$ with $I \neq R$. Let n be a positive integer and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. In [7], the authors defined I to be a strongly α-n-irreducible ideal if whenever $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap I_2 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$ for $I_1, I_2, \ldots, I_{n+1}$ ideals of R, there are n of the I_i's whose intersection is in I. As was done in [5], the main purpose of this paper is to generalize the last concept to the context of graded rings.

In section 2 we show, among other things, that the concepts of strongly homogeneous α-n-irreducible ideals and of strongly homogeneous n-irreducible ideals are different in general, and that the concepts of strongly α-n-irreducible ideals and of strongly homogeneous α-n-irreducible ideals are different in general, see Example 2.3 and Example 2.4.

Before we begin our study, we will first recall some basic properties and terminology related to graded ring theory. Unless otherwise stated, M will denote a commutative additive monoid with identity element denoted by 0. Let R be a graded ring. If I is a homogeneous ideal of a graded ring R, then R/I is a graded ring, where $(R/I)_\alpha := (R_\alpha + I)/I$. Suppose that A, B are graded rings and $R = A \times B$. Then R is a graded ring by $R_g = A_g \times B_g$ for all $g \in G$. Also, it is easy to see that an ideal I of R is a homogeneous ideal if and only if $I = J \times K$ for some homogeneous ideals J of A and K of B. Let R be a graded ring and I, J be homogeneous ideals of R. Then, it is well known that $I + J$, IJ and $I \cap J$ are homogeneous ideals of R.

Let G be a group, R be a graded ring, and S be a multiplicative set of homogeneous elements of R. Then $S^{-1}R$ is a graded ring by $\left(S^{-1}R\right)_g =$

$\left\{\frac{a}{s} \mid a \in R_h,\ s \in S \cap R_{h-g}\right\}$ for all $g \in G$. If I is a homogeneous ideal of R, then it is easy to see that $S^{-1}I$ is a homogeneous ideal of $S^{-1}R$.

Let R be a graded ring and E be an R-module. By a graded R-module E, we mean an R-module graded by M, that is, a direct sum of subgroups E_α of E such that $R_\alpha E_\beta \subseteq E_{\alpha+\beta}$ for every $\alpha, \beta \in M$. The set $h(E) = \bigcup_{\alpha \in G} E_\alpha$ is the set of homogeneous elements of E. A submodule N of E is said to be graded if $N = \bigoplus_{\alpha \in M}(N \cap E_\alpha)$, equivalently, if N is generated by homogeneous elements. In the literature, graded submodules and homogeneous submodules are also used interchangeably, but graded submodules seem to be preferred, so we will use that term in this paper.

Let R and R' be two graded rings. A ring homomorphism $f : R \to R'$ is said to be graded if $f(R_\alpha) \subseteq R'_\alpha$ for all $\alpha \in G$. A graded ring isomorphism is a bijective graded ring homomorphism. For more information and other terminology on graded rings and modules, we refer the reader to [16] and [17].

2. Strongly homogeneous α-n-irreducible ideals

We begin with the following definition.

Definition 2.1. Let R be a graded ring, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function, where $\mathcal{L}(R)$ is the set of all homogeneous ideals of R.. A proper homogeneous ideal I of R is called a strongly homogeneous α-n-irreducible ideal if whenever $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$ for $I_1, \ldots, I_{n+1}$ homogeneous ideals of R, there are n of the I_i's whose intersection is in I, without loss of generality, we may assume that $I_1 \cap \cdots \cap I_n \subseteq I$.

Remark 2.2. It is clear that every strongly homogeneous α-n-irreducible ideal of a graded ring R is a strongly homogeneous n-irreducible ideal of R and that every strongly α-n-irreducible homogeneous ideal of a graded ring R is a strongly homogeneous α-n-irreducible ideal of R. In Example 2.3 and Example 2.4, we show that the two implications are in general not reversible. It is also clear that if $\alpha(0) \neq \emptyset$, then the zero ideal is a strongly homogeneous α-n-irreducible ideal of R for each positive integer $n \geq 1$. Note that a strongly homogeneous α-irreducible ideal is just a strongly homogeneous α-1-irreducible ideal.

The following implications summarize the relationship between the above concepts.

strongly homogeneous irred. $\Rightarrow$ strongly homogeneous n-irred.
$\Rightarrow$ strongly homogeneous α-n-irred.

Next, we give an example of a strongly homogeneous α-n-irreducible ideal which is not strongly homogeneous n-irreducible, and an example of a strongly homogeneous α-2-irreducible ideal which is not strongly α-2-irreducible.

Example 2.3. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers with its natural $\mathbb{Z}_2$-grading. Let $I = p_1 p_2 \cdots p_{n+1} R$, where $p_1, p_2, \ldots, p_{n+1}$ are prime homogeneous elements of R. By [5, Theorem 3.5] I is not a graded n-irreducible ideal of R, so it is not strongly homogeneous n-irreducible for any positive integer n. By taking $\alpha(I) = I$, I is a strongly homogeneous α-n-irreducible ideal.

Example 2.4. Consider the ideal $I = \langle 6 \rangle$ generated by the homogeneous element 6 of the ring $\mathbb{Z}[i]$ of Gaussian integers, which is a graded ring with $G = \mathbb{Z}_2$. Set $\alpha(I) = \emptyset$. By [5, Theorem 3.5], I is homogeneous strongly α-2-irreducible. But, by [20, Theorem 3.1], I is not strongly α-2-irreducible.

Throughout this paper, we consider $\alpha(I) \subsetneq I$ for each homogeneous ideal I of a graded ring R. Let $\alpha, \beta : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be two functions. If α and β satisfy $\alpha(I) \subseteq \beta(I)$ for every $I \in \mathcal{L}(R)$, the last inclusion is denoted by $\alpha \leq \beta$ in the following result.

Proposition 2.5. *Let R be a graded ring, I be a homogeneous ideal of R, n be a positive integer and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function.*

1. *If I is strongly homogeneous α-n-irreducible, then I is strongly homogeneous β-n-irreducible for each function $\beta : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ such that $\alpha \leq \beta$.*
2. *If I is strongly homogeneous α-n-irreducible, then I is strongly homogeneous α-m-irreducible for each positive integer $m > n$.*
3. *If I is strongly homogeneous α-n-irreducible for some $n \geq 1$, then there exists the smallest integer $n_0 \geq 1$ such that I is strongly homogeneous α-n_0-irreducible. In this case, I is strongly homogeneous α-n-irreducible for all $n \geq n_0$ and it is not strongly homogeneous α-m-irreducible for $n_0 > m > 0$.*

Proof. (1) Assume that I is strongly homogeneous α-n-irreducible and let $\beta : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function such that $\alpha \leq \beta$. Consider $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \beta(I)$ for $I_1, \ldots, I_{n+1}$ homogeneous ideals of R. Obviously, $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$. By hypothesis, there are n of the I_i's whose intersection is in I. So we can assume that $I_1 \cap \cdots \cap I_n \subseteq I$. Thus I is a strongly homogeneous β-n-irreducible ideal.

(2) Assuming that I is strongly homogeneous α-n-irreducible, we prove that I is strongly homogeneous α-m-irreducible for any positive integer $m \geq n$. Indeed, let $m > n$, $I_1 \cap \cdots \cap I_{m+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{m+1} \nsubseteq \alpha(I)$ for $I_1, \ldots, I_{m+1}$ homogeneous ideals of R. Consider $J_i = I_i$ for each $i = 1, \ldots, n$, and $J_{n+1} = I_{n+1} \cap \cdots \cap I_{m+1}$, so $J_1 \cap \cdots \cap J_{n+1} \subseteq I$ and $J_1 \cap \cdots \cap J_{n+1} \nsubseteq \alpha(I)$. By our assumption, there are n of the J_i's whose intersection is in I, so we can assume that $J_1 \cap \cdots \cap J_n \subseteq I$. Hence $I_1 \cap \cdots \cap I_m \subseteq J_1 \cap \cdots \cap J_n \subseteq I$, as desired.

(3) This is easy. □

The next result gives a sufficient condition for the equivalence between the two concepts of strongly homogeneous irreducible ideals and strongly homogeneous α-irreducible ideals.

Theorem 2.6. *Let R be a graded ring, $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function, and I be a proper homogeneous ideal of R. If I is a strongly homogeneous α-irreducible ideal that is not strongly homogeneous irreducible, then $I^2 \subseteq \alpha(I)$. So every strongly homogeneous α-irreducible ideal I with $I^2 \nsubseteq \alpha(I)$ is strongly homogeneous irreducible.*

Proof. Suppose that I is strongly homogeneous α-irreducible with $I^2 \nsubseteq \alpha(I)$. We prove that I is strongly homogeneous irreducible. Indeed, let $I_1 \cap I_2 \subseteq I$, where I_1 and I_2 are two homogeneous ideals of R. If $I_1 \cap I_2 \nsubseteq \alpha(I)$, then $I_1 \subseteq I$ or $I_2 \subseteq I$ by hypothesis. Thus we may assume that $I_1 \cap I_2 \subseteq \alpha(I)$. If $I_1 \cap I^2 \nsubseteq \alpha(I)$, then $I_1 \cap (I_2 + I^2) \subseteq I$ and $I_1 \cap (I_2 + I^2) \nsubseteq \alpha(I)$, since $(I_1 \cap I_2) + (I_1 \cap I^2) \subseteq I_1 \cap (I_2 + I^2)$. Hence $I_1 \subseteq I$ or $I_2 + I^2 \subseteq I$, so $I_1 \subseteq I$ or $I_2 \subseteq I$. Thus, we can assume that $I_1 \cap I^2 \subseteq \alpha(I)$. Similarly, we assume that $I_2 \cap I^2 \subseteq \alpha(I)$. Since $(I_1 \cap I_2) + (I_1 \cap I^2) + (I_2 \cap I^2) + I^2 \subseteq (I_1 + I^2) \cap (I_2 + I^2)$, we have $(I_1 + I^2) \cap (I_2 + I^2) \subseteq I$ and $(I_1 + I^2) \cap (I_2 + I^2) \nsubseteq \alpha(I)$. Thus $I_1 + I^2 \subseteq I$ or $I_2 + I^2 \subseteq I$, so $I_1 \subseteq I$ or $I_2 \subseteq I$. □

The following corollary is a direct consequence of the previous theorem.

Corollary 2.7. *Let R be a graded ring, n be a positive integer, $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function, and I be a proper homogeneous ideal of R. If I is a strongly homogeneous α-n-irreducible ideal that is not strongly homogeneous n-irreducible, then $I^{n+1} \subseteq \alpha(I)$. So every strongly homogeneous α-n-irreducible ideal I with $I^{n+1} \nsubseteq \alpha(I)$ is strongly homogeneous n-irreducible.*

Proposition 2.8. *Let R be a graded ring and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function such that $\alpha(J) \subseteq \alpha(I)$ for each $I \subseteq J$ homogeneous ideals of R. If I_i is a strongly homogeneous α-n_i-irreducible ideal of R for every $1 \leq i \leq m$, then $I_1 \cap \cdots \cap I_m$ is a strongly homogeneous α-n-irreducible ideal of R for $n = n_1 + \cdots + n_m$.*

Proof. Recall from the introduction that any intersection of homogeneous ideals is a homogeneous ideal. By induction on m, it suffices to prove the result for $m = 2$. Assume that I_1 is a strongly homogeneous α-n_1-irreducible ideal of R and I_2 is a strongly homogeneous α-n_2-irreducible ideal of R. Take $I = I_1 \cap I_2$ and $n = n_1 + n_2$. Let $J_1 \cap \cdots \cap J_{n+1} \subseteq I$ and $J_1 \cap \cdots \cap J_{n+1} \nsubseteq \alpha(I)$. Obviously, $J_1 \cap \cdots \cap J_{n+1} \subseteq I_1$ and $J_1 \cap \cdots \cap J_{n+1} \nsubseteq \alpha(I_1)$; $J_1 \cap \cdots \cap J_{n+1} \subseteq I_2$ and $J_1 \cap \cdots \cap J_{n+1} \nsubseteq \alpha(I_2)$. By our assumption, there are n of the J_j's whose intersection is in I_1 (respectively, I_2), without loss of generality, we may assume that $J_1 \cap \cdots \cap J_{n_1} \subseteq I_1$ (respectively, $J_{n_1+1} \cap \cdots \cap J_{n_1+n_2} \subseteq I_2$). Hence, $J_1 \cap \cdots \cap J_n \subseteq I$. □

Theorem 2.9. *Let $f : R \to S$ be a surjective graded ring homomorphism, $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ and $\beta : \mathcal{L}(S) \to \mathcal{L}(S) \cup \{\emptyset\}$ be two functions satisfying $f(\alpha(I)) \subseteq \beta(f(I))$ for each homogeneous ideal I of R with $\alpha(I) \neq \emptyset$, and $\beta(f(I)) = \emptyset$ if $\alpha(I) = \emptyset$. Let I be a homogeneous ideal of R. If $f(I) \cap R$ is a strongly homogeneous α-n-irreducible ideal of R, then $f(I)$ is a strongly homogeneous β-n-irreducible ideal of S.*

Proof. Since f is surjective, we have that $f(J \cap R) = J$ for every ideal J of S. Since f is a graded ring homomorphism, $f(I)$ is a homogeneous ideal of S. Assume that $f(I) \cap R$ is a strongly homogeneous α-n-irreducible ideal of R, where $\alpha(I) \neq \emptyset$ and assume that $J_1 \cap \cdots \cap J_{n+1} \subseteq f(I)$ and $J_1 \cap \cdots \cap J_{n+1} \nsubseteq \beta(f(I))$ for $J_1, J_2, \ldots, J_{n+1}$ homogeneous ideals of S. Then we have that $(J_1 \cap R) \cap \cdots \cap (J_{n+1} \cap R) \subseteq f(I) \cap R$ and we prove that $(J_1 \cap R) \cap \cdots \cap (J_{n+1} \cap R) \nsubseteq \alpha(f(I) \cap R)$. Suppose the contrary and let $y \in J_1 \cap \cdots \cap J_{n+1}$. since f is surjective, there exists $x \in J_1 \cap \cdots \cap J_{n+1} \cap R$ such that $y = f(x)$, hence $x \in \alpha(f(I) \cap R)$. So $y \in f(\alpha(f(I) \cap R)) = \beta(f(f(I) \cap R)) = \beta(f(I))$, a contradiction. By hypothesis, we obtain $(J_1 \cap R) \cap \cdots \cap (J_n \cap R) \subseteq f(I) \cap R$. Therefore, $f((J_1 \cap R) \cap \cdots \cap (J_n \cap R)) \subseteq f(f(I) \cap R)$, and so $J_1 \cap \cdots \cap J_n \subseteq f(I)$, which satisfies the proof. For the other case: $\alpha(I) = \emptyset$, we need to prove that $f(I)$ is strongly homogeneous n-irreducible, which can be done in the same way. □

Remark 2.10. If I is strongly homogeneous α-n-irreducible and $\ker(f) \subseteq I$, then by Theorem 2.9, $f(I)$ is a strongly homogeneous β-n-irreducible ideal, since $f(I) \cap R = I$.

Let J be a homogeneous ideal of R and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. We define $\alpha_J : \mathcal{L}(R/J) \to \mathcal{L}(R/J) \cup \{\emptyset\}$ by $\alpha_J(I/J) = (\alpha(I) + J)/J$ for every homogeneous ideal $I \in \mathcal{L}(R)$ containing J with $\alpha(I) \neq \emptyset$, and $\alpha_J(I/J) = \emptyset$ if $\alpha(I) = \emptyset$. Obviously α_J is also a function.

Corollary 2.11. *Let I, J be two homogeneous ideals of a graded ring R such that $J \subseteq I$, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. Then the following statements hold:*

(1) If I is a homogeneous strongly α-n-irreducible ideal of R, then I/J is a strongly homogeneous α_J-n-irreducible ideal of R/J.

(2) Assume that $J \subseteq \alpha(I)$. If I/J is a strongly homogeneous α_J-n-irreducible ideal of R/J, then I is a strongly homogeneous α-n-irreducible ideal of R.

Proof. (1) This follows directly from Theorem 2.9.

(2) Let $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$ for some $I_1, \ldots, I_{n+1}$ homogeneous ideals of R. Then $I_1/J \cap \cdots \cap I_{n+1}/J \subseteq I/J$ and $I_1/J \cap \cdots \cap I_{n+1}/J \nsubseteq \alpha(I)/J = \alpha_J(I/J)$, since $J \subseteq \alpha(I)$. By hypothesis, I/J is a strongly homogeneous α_J-n-irreducible ideal. Thus, without loss of generality, $I_1/J \cap \cdots \cap I_n/J \subseteq I/J$, and so $I_1 \cap \cdots \cap I_n \subseteq I$.

3. Strongly homogeneous α-n-irreducible ideals in trivial graded ring extension and the amalgamation of graded rings along a homogeneous ideal

In this section we study the transfer of the strongly α-n-irreducible property in some graded ring theoretic constructions. For a graded ring R and a graded R-module E, the set of all graded submodules of E is denoted by $\mathcal{S}(E)$.

Definition 3.1. Let R be a graded ring, F be a graded submodule of a graded R-module E, n be a positive integer, and $\beta : \mathcal{S}(E) \to \mathcal{S}(E) \cup \{\emptyset\}$ be a function, where $\mathcal{S}(E)$ is the set of all graded submodules of E. Then F is said to be strongly graded β-n-irreducible if whenever $F_1 \cap \cdots \cap F_{n+1} \subseteq F$ and $F_1 \cap \cdots \cap F_{n+1} \nsubseteq \beta(F)$ for some $F_1, \ldots, F_{n+1}$ graded submodules of E, there are n of the F_i's whose intersection is in F. Without loss of generality, we may assume that $F_1 \cap \cdots \cap F_n \subseteq F$. Also, we may assume that $\beta(F) \subseteq F$.

Let R be a ring and E be an R-module. The following ring construction, called the trivial ring extension of R by E (also called the idealization of E), was introduced by Nagata [15, page 2]. It is the set of pairs (r, e) with pairwise addition and multiplication given by $(r, e)(q, f) = (rq, rf + qe)$, denoted by $R \propto E$, whose underlying abelian group is $A \times E$. This construction has been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. For more information the reader is referred to [1, 6, 10, 13, 14]. Now, if R is a graded ring and E is a graded R-module, then $R \propto E$ is a graded ring by $(R \propto E)_g = R_g \oplus E_g$ for all $g \in G$, see [2]. Recently, many researchers have studied the transfer of various homogeneous properties in the idealization of graded modules, see for example [3, 4, 19]. Let I be an ideal of R and F be a submodule of E. Then by [2, Theorem 1], $I \propto F$ is a homogeneous ideal of $R \propto E$ if and only if I is a homogeneous ideal of R and F is a graded submodule of E and $IE \subseteq F$. For a homogeneous ideal H of the trivial graded ring extension of a graded ring R by a graded R-module E, we set $I_H = \{a \in R \mid (a, e) \in H \text{ for some } e \in E\}$ and $F_H = \{e \in E \mid (a, e) \in H \text{ for some } a \in R\}$. Note that I_H is a homogeneous ideal and F_H is a graded submodule, a consequence of the fact that H is a homogeneous ideal.

In this section we first study the extension of strongly homogeneous α-n-irreducible ideals to the trivial graded ring extension.

Proposition 3.2. *Let R be a graded ring, E be a graded R-module, F be a graded submodule of E, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ and $\beta : \mathcal{S}(E) \to \mathcal{S}(E) \cup \{\emptyset\}$ be two functions with $\alpha(0) \in \{\emptyset, 0\}$. Let $\phi : \mathcal{L}(R \propto E) \to \mathcal{L}(R \propto E) \cup \{\emptyset\}$ be a function satisfying:*

$$\phi(0 \propto F) = \begin{cases} 0 \propto \beta(F) & \text{if } \alpha(0) = 0 \\ \emptyset & \text{if } \alpha(0) = \emptyset \end{cases}$$

Then:

(1) If $0 \propto F$ is a strongly homogeneous ϕ-n-irreducible ideal, then F is a strongly β-n-irreducible submodule of E.

(2) If R is a graded domain, E is a divisible graded module, and $\alpha(0) = 0$, then $0 \propto F$ is strongly homogeneous ϕ-n-irreducible if and only if F is a strongly graded β-n-irreducible submodule of E.

Proof. (1) Let $F_1 \cap \cdots \cap F_{n+1} \subseteq F$ and $F_1 \cap \cdots \cap F_{n+1} \nsubseteq \beta(F)$ for some graded submodules $F_1, \ldots, F_{n+1}$ of E. Then $(0 \propto F_1) \cap \cdots \cap (0 \propto F_{n+1}) \subseteq 0 \propto F$ and $(0 \propto F_1) \cap \cdots \cap (0 \propto F_{n+1}) \nsubseteq \phi(0 \propto F)$. By hypothesis, we get that $(0 \propto F_1) \cap \cdots \cap (0 \propto F_n) \subseteq 0 \propto F$, and so $F_1 \cap \cdots \cap F_n \subseteq F$, as desired.

(2) The "only if" follows from the previous statement. Conversely, let $H_1 \cap \cdots \cap H_{n+1} \subseteq 0 \propto F$ and $H_1 \cap \cdots \cap H_{n+1} \nsubseteq \phi(0 \propto F)$. If there exists i such that $I_{H_i} \neq 0$, then $H_i = I_{H_i} \propto E$. So it iss clear that $H_1 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_{n+1} = H_1 \cap \cdots \cap H_{n+1} \subseteq 0 \propto F$. Now, if for all $i = 1, \ldots, n+1$ we have $I_{H_i} = 0$, then $H_i = 0 \propto F_{H_i}$, so $(0 \propto F_{H_1}) \cap \cdots \cap (0 \propto F_{H_{n+1}}) \subseteq 0 \propto F$ and $(0 \propto F_{H_1}) \cap \cdots \cap (0 \propto F_{H_{n+1}}) \nsubseteq 0 \propto \beta(F)$. Hence $F_{H_1} \cap \cdots \cap F_{H_{n+1}} \subseteq F$ and $F_{H_1} \cap \cdots \cap F_{H_{n+1}} \nsubseteq \beta(F)$, so we get that $F_{H_1} \cap \cdots \cap F_{H_n} \subseteq F$. Hence $(0 \propto F_{H_1}) \cap \cdots \cap (0 \propto F_{H_n}) \subseteq 0 \propto F$. □

Example 3.3. Let $R = \mathbb{Z}[i]$ with its natural $\mathbb{Z}_2$-grading, and $I = p_1 p_2 \cdots p_{n+1} R$, where $p_1, p_2, \ldots, p_{n+1}$ are some prime homogeneous elements of R. We consider the function $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ satisfying $\alpha(0) = \emptyset$ and $R \propto R$ the trivial graded ring extension. We set $\phi : \mathcal{L}(R \propto R) \to \mathcal{L}(R \propto R) \cup \{\emptyset\}$ to be a function satisfying $\phi(0 \propto J) = \emptyset$ for all $J \in \mathcal{L}(R)$. By Example 2.3,

I is not a strongly homogeneous n-irreducible ideal. Hence $0 \propto F$ is not strongly homogeneous n-irreducible.

Theorem 3.4. *Let R be a graded ring, E be a graded R-module, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. Let $\beta : \mathcal{L}(R \propto E) \to \mathcal{L}(R \propto E) \cup \{\emptyset\}$ be a function satisfing:*

$$\beta(I \propto F) = \begin{cases} \alpha(I) \propto F & \text{if } \alpha(I) \neq \emptyset \\ \emptyset & \text{if } \alpha(I) = \emptyset \end{cases}$$

where I is a homogeneous ideal of R and F is a graded submodule of E satisfying $IE \subseteq F$. Then:

(1) $I \propto E$ is a strongly homogeneous β-n-irreducible ideal of $R \propto E$ if and only if I is a strongly homogeneous α-n-irreducible ideal of R.
(2) If I is a strongly homogeneous α-n_1-irreducible ideal of R and F is a strongly homogeneous n_2-irreducible submodule of E, then $I \propto F$ is a strongly homogeneous β-n-irreducible ideal of $R \propto E$, where $n = n_1 + n_2$.

Proof. (1) For "only if" assume that $I \propto E$ is a strongly homogeneous β-n-irreducible ideal of $R \propto E$ and let $I_1, \ldots, I_{n+1}$ be homogeneous ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$. So $(I_1 \propto E) \cap \cdots \cap (I_{n+1} \propto E) \subseteq I \propto E$ and $(I_1 \propto E) \cap \cdots \cap (I_{n+1} \propto E) \nsubseteq \beta(I \propto E)$. By hypothesis, we get $(I_1 \propto E) \cap \cdots \cap (I_n \propto E) \subseteq I \propto E$, and so $I_1 \cap \cdots \cap I_n \subseteq I$. Conversely, assume that I is a strongly homogeneous α-n-irreducible ideal of R. Let $H_1 \cap \cdots \cap H_{n+1} \subseteq I \propto E$ and $H_1 \cap \cdots \cap H_{n+1} \nsubseteq \beta(I \propto E)$, where $H_1, \ldots, H_{n+1}$ are homogeneous ideals of $R \propto E$. Then $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \subseteq I$ and $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \nsubseteq \alpha(I)$. So by our assumption we get that $I_{H_1} \cap \cdots \cap I_{H_n} \subseteq I$, and so $H_1 \cap \cdots \cap H_n \subseteq (I_{H_1} \propto E) \cap \cdots \cap (I_{H_n} \propto E) \subseteq I \propto E$.

(2) Assume that I is a strongly homogeneous α-n_1-irreducible ideal of R and F is a strongly graded n_2-irreducible submodule of E. Set $n = n_1 + n_2$ and let $H_1 \cap \cdots \cap H_{n+1} \subseteq I \propto F$ and $H_1 \cap \cdots \cap H_{n+1} \nsubseteq \beta(I \propto F)$, where $H_1, \ldots, H_{n+1}$ are homogeneous ideals of $R \propto E$. Then $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \subseteq I$ and $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \nsubseteq \alpha(I)$ and $F_{H_1} \cap \cdots \cap F_{H_{n+1}} \subseteq F$. Thus by our assumption we get $I_{H_1} \cap \cdots \cap I_{H_{n_1}} \subseteq I$ and $F_{H_{n_1+1}} \cap \cdots \cap F_{H_n} \subseteq F$. So it is clear that $H_1 \cap \cdots \cap H_n \subseteq I \propto F$. □

Let R and S be two rings, J be an ideal of S, and $f : R \to S$ be a ring homomorphism. The following ring construction, called the amalgamation of R with S along J with respect to f, is a subring of $R \times S$ defined by:

$$R \bowtie^f J := \{(r, f(r) + j) \mid r \in R, j \in J\}$$

This construction was introduced and studied in [8]. Let R and S be two graded rings, J be a homogeneous ideal of S, and $f : R \to S$ be a graded ring homomorphism. Then $R \bowtie^f J$ is a graded ring by $(R \bowtie^f J)_g = \{(r_g, f(r_g) + j_g) \mid r_g \in R_g, j \in J_g\}$ for all $g \in G$, see [9,11]. Let I be an ideal of R. Then $I \bowtie^f J$ is a homogeneous ideal of $R \bowtie^f J$ if and only if I is a homogeneous ideal of R, see [11, Theorem 3.5(2)]. Let H be a homogeneous ideal of $R \bowtie^f J$ and set $I_H = \{a \in R \mid (a, f(a) + j) \in H \text{ for some } j \in J\}$ and $J_H = \{j \in J \mid (a, f(a) + j) \in H \text{ for some } a \in R\}$. Note that I_H and J_H are two homogeneous ideals, a consequence of the fact that H is a homogeneous ideal.

The following theorem examines the transfer of strongly homogeneous α-n-irreducible ideals in the amalgamation of graded rings.

Theorem 3.5. *Let R and S be two graded rings, $f : R \to S$ be a graded ring homomorphism, J be a homogeneous ideal of S, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. Let $\beta : \mathcal{L}(R \bowtie^f J) \to \mathcal{L}(R \bowtie^f J) \cup \{\emptyset\}$ be a function satisfying:*

$$\beta(I \bowtie^f K) = \begin{cases} \alpha(I) \bowtie^f K & \text{if } \alpha(I) \neq \emptyset \\ \emptyset & \text{if } \alpha(I) = \emptyset \end{cases}$$

and for $I \subseteq f^{-1}(J)$, we have:

$$\beta(I \bowtie^f 0) = \begin{cases} \alpha(I) \bowtie^f & \text{if } \alpha(I) \neq \emptyset \\ \emptyset & \text{if } \alpha(I) = \emptyset \end{cases}$$

where I is a homogeneous ideal of R. Then:

(1) $I \bowtie^f J$ is a strongly homogeneous β-n-irreducible ideal of $R \bowtie^f J$ if and only if I is a strongly homogeneous α-n-irreducible ideal of R.

(2) Suppose that $f(I)J \subseteq K$, where K is a homogeneous ideal contained in J. If I is a strongly homogeneous α-n_1-irreducible ideal of R and K is strongly homogeneous n_2-irreducible, then $I \bowtie^f K$ is a strongly homogeneous β-n-irreducible ideal of $R \bowtie^f J$, where $n = n_1 + n_2$.

(3) Suppose that $I \subseteq f^{-1}(J)$. If I is a strongly homogeneous α-n_1-irreducible ideal of R and the zero ideal of S is strongly homogeneous n_2-irreducible, then $I \times 0$ is a strongly homogeneous β-n-irreducible ideal of $R \bowtie^f J$, where $n = n_1 + n_2$.

Proof. (1) Suppose that $I \bowtie^f J$ is a strongly homogeneous β-n-irreducible ideal of $R \bowtie^f J$ and let $I_1, \ldots, I_{n+1}$ be homogeneous ideals of R satisfying $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I)$. Then $(I_1 \bowtie^f J) \cap \cdots \cap (I_{n+1} \bowtie^f J) \subseteq I \bowtie^f J$ and $(I_1 \bowtie^f J) \cap \cdots \cap (I_{n+1} \bowtie^f J) \nsubseteq \beta(I \bowtie^f J)$. By hypothesis, we get $(I_1 \bowtie^f J) \cap \cdots \cap (I_n \bowtie^f J) \subseteq I \bowtie^f J$, and so we get that $I_1 \cap \cdots \cap I_n \subseteq I$. Conversely, suppose that I is homogeneous strongly α-n-irreducible and let $H_1 \cap \cdots \cap H_{n+1} \subseteq I \bowtie^f J$ and $H_1 \cap \cdots \cap H_{n+1} \nsubseteq \beta(I \bowtie^f J)$. Obviously, $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \subseteq I$ and $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \nsubseteq \alpha(I)$. Hence, by hypothesis, we obtain $I_{H_1} \cap \cdots \cap I_{H_n} \subseteq I$, and so $H_1 \cap \cdots \cap H_n \subseteq (I_{H_1} \bowtie^f J) \cap \cdots \cap (I_{H_n} \bowtie^f J) \subseteq I \bowtie^f J$.

(2) Suppose I is a homogeneous strongly α-n_1-irreducible ideal of R and K is a strongly homogeneous n_2-irreducible ideal of S. Let $H_1 \cap \cdots \cap H_{n+1} \subseteq I \bowtie^f K$ and $H_1 \cap \cdots \cap H_{n+1} \nsubseteq \beta(I \bowtie^f K)$, where $n = n_1 + n_2$ and $H_1, \ldots, H_{n+1}$ are homogeneous ideals of $R \bowtie^f J$. Then $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \subseteq I$ and $I_{H_1} \cap \cdots \cap I_{H_{n+1}} \nsubseteq \alpha(I)$ and $J_{H_1} \cap \cdots \cap J_{H_{n+1}} \subseteq K$. So by our assumption we get $I_{H_1} \cap \cdots \cap I_{H_{n_1}} \subseteq I$ and $J_{H_{n_1+1}} \cap \cdots \cap J_{H_n} \subseteq K$. We can easily prove that $H_1 \cap \cdots \cap H_{n_1+n_2} \subseteq I \bowtie^f K$.

(3) This is similar to the proof of (2). □

Let G be an abelian group. Consider a graded ring R. It is well known that if we take a multiplicative set S consisting only of homogeneous elements of R, then $S^{-1}R$ is a graded ring. Let $f : R \to S^{-1}R$ be the natural graded ring homomorphism defined by $f(r) = \frac{r}{1}$. For each homogeneous ideal I of the graded ring $S^{-1}R$, we consider $I^c = \{r \in R \mid \frac{r}{1} \in I\} = I \cap R$, which is a homogeneous ideal of R, and $C^h = \{I^c \mid I$ is a homogeneous ideal of $S^{-1}R\}$.

Proposition 3.6. *Let G be an abelian group, R be a graded ring, S be a multiplicative set of homogeneous elements of R, n be a positive integer, and $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function. Let $\beta : \mathcal{L}(S^{-1}R) \to \mathcal{L}(S^{-1}R) \cup \{\emptyset\}$ be a function satisfying:*

$$\beta(S^{-1}I) = \begin{cases} S^{-1}(\alpha(I)) & \text{if } \alpha(I) \neq \emptyset \\ \emptyset & \text{if } \alpha(I) = \emptyset \end{cases}$$

and $(\beta(I))^c = \alpha(I^c)$.

Then there is a one-to-one correspondence between the strongly homogeneous β-n-irreducible ideals of $S^{-1}R$ and the strongly homogeneous α-n-irreducible ideals of R contained in C^h which do not meet S.

Proof. Suppose that I is a homogeneous strongly β-n-irreducible ideal of $S^{-1}R$. Let $I_1, \dots, I_{n+1}$ be homogeneous ideals of R such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I^c$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \alpha(I^c) = (\beta(I))^c$. Then $(S^{-1}I_1) \cap \cdots \cap (S^{-1}I_{n+1}) = S^{-1}(I_1 \cap \cdots \cap I_{n+1}) \subseteq S^{-1}(I^c) = I$. Obviously, $(S^{-1}I_1) \cap \cdots \cap (S^{-1}I_{n+1}) \nsubseteq S^{-1}\alpha(I^c) = \beta(S^{-1}I^c) = \beta(I)$. Suppose the contrary and let $r \in I_1 \cap \cdots \cap I_{n+1}$. Then $\frac{r}{1} \in (S^{-1}I_1) \cap \cdots \cap (S^{-1}I_{n+1}) \subseteq \beta(I)$. So $r \in (\beta(I))^c$, a contradiction. Since I is strongly homogeneous β-n-irreducible, $(S^{-1}I_1) \cap \cdots \cap (S^{-1}I_n) \subseteq I$. So $I_1 \cap \cdots \cap I_n \subseteq I^c$. Therefore, I^c is a strongly α-n-irreducible ideal of R.

Conversely, let I be a strongly homogeneous α-n-irreducible ideal of R such that $I \cap S = \emptyset$, so $S^{-1}I \neq S^{-1}R$. Let $I_1 \cap \cdots \cap I_{n+1} \subseteq S^{-1}I$ and $I_1 \cap \cdots \cap I_{n+1} \nsubseteq \beta(S^{-1}I)$, where $I_1, \dots, I_{n+1}$ are homogeneous ideals of $S^{-1}R$. Hence $(I_1)^c \cap \cdots \cap (I_{n+1})^c = (I_1 \cap \cdots \cap I_{n+1})^c \subseteq (S^{-1}I)^c$ and $(I_1^c) \cap \cdots \cap (I_{n+1})^c \nsubseteq (\beta(S^{-1}I))^c$. Suppose the contrary, let $\frac{r}{t} \in I_1 \cap \cdots \cap I_{n+1}$. Then $\frac{r}{t} = \frac{u}{s}$ where $u \in I_1^c \cap \cdots \cap I_{n+1}^c \subseteq (\beta(S^{-1}I))^c$. Since $I \in C^h$, we have $(S^{-1}I)^c = I$. Then $(\beta(S^{-1}I))^c = \alpha(I)$, so $\frac{r}{t} \in S^{-1}(\alpha(I)) = \beta(S^{-1}I)$, a contradiction. So $(I_1)^c \cap \cdots \cap (I_{n+1})^c \subseteq I$ and $(I_1)^c \cap \cdots \cap (I_{n+1})^c \nsubseteq \alpha(I)$. Hence, $(I_1)^c \cap \cdots \cap (I_n)^c \subseteq I$. Therefore, $I_1 \cap \cdots \cap I_n = S^{-1}((I_1)^c) \cap \cdots \cap S^{-1}((I_n)^c) \subseteq S^{-1}I$. As a result, $S^{-1}I$ is a strongly β-n-irreducible ideal of $S^{-1}R$. □

Now we examine the strongly homogeneous α-n-irreducible ideals in the product of two, and hence any finite number of graded rings.

Proposition 3.7. *Let I and J be homogeneous ideals of a graded ring R_1 and R_2 respectively and set $R := R_1 \times R_2$. Let $\alpha_i : \mathcal{L}(R_i) \to \mathcal{L}(R_i) \cup \{\emptyset\}$ be a function for each $i = 1, 2$. Let $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function satisfying:*

$$\alpha(I \times J) = \begin{cases} \alpha_1(I) \times \alpha_2(J) & \text{if } \alpha_1(I) \neq \emptyset \text{ and } \alpha_2(J) \neq \emptyset \\ \emptyset & \text{if } \alpha_1(I) = \emptyset \text{ or } \alpha_2(J) = \emptyset \end{cases}$$

Then $I \times J$ is a strongly homogeneous α-n-irreducible ideal of R if and only if I and J are strongly homogeneous n-irreducible ideals of R_1 and R_2, respectively.

Proof. Assume that $I \times J$ is a strongly homogeneous α-n-irreducible ideal of R. We show that I (respectively, J) is a strongly homogeneous n-irreducible ideal of R_1 (respectively, R_2). Suppose on the contrary that I is not strongly homogeneous n-irreducible. Then there are $n+1$ homogeneous ideals $I_1, \ldots, I_{n+1}$ of R_1 such that $I_1 \cap \cdots \cap I_{n+1} \subseteq I$ and any intersection of n homogeneous ideals among these homogeneous ideals is not in I. Hence, $(I_1 \times J) \cap \cdots \cap (I_{n+1} \times J) \subseteq I \times J$ and $(I_1 \times J) \cap \cdots \cap (I_{n+1} \times J) \nsubseteq \alpha(I \times J)$, since $\alpha_2(J) \subsetneq J$. Since $I \times J$ is strongly homogeneous α-n-irreducible, $(I_1 \times J) \cap \cdots \cap (I_n \times J) \subseteq I \times J$, a contradiction. So I is a strongly homogeneous n-irreducible ideal of R_1. Similarly, we claim that J is a strongly homogeneous n-irreducible ideal of R_2. The necessity holds by [5, Proposition 2.22]. □

The following result is a direct consequence of the previous proposition.

Corollary 3.8. *Let I_i be a homogeneous ideal of a graded ring R_i and $\alpha_i : \mathcal{L}(R_i) \to \mathcal{L}(R_i) \cup \{\emptyset\}$ be a function for each $1 \leq i \leq m$. Consider $R = R_1 \times \cdots \times R_m$. Let $\alpha : \mathcal{L}(R) \to \mathcal{L}(R) \cup \{\emptyset\}$ be a function satisfying:*

$$\alpha(I_1 \times ... \times I_m) = \begin{cases} \prod_1^m \alpha_i(I_i) & \text{if every } \alpha_i(I_i) \neq \emptyset \\ \emptyset & \text{if } \alpha(I_i) = \emptyset \text{ for some } i = 1, \ldots, m \end{cases}$$

Then $\prod_1^m I_i$ is a strongly homogeneous α-n-irreducible ideal of R if and only if I_i is a strongly homogeneous n-irreducible ideal of R_i for each $1 \leq i \leq m$.

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